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Exact solutions for functionally graded anisotropic cylinders subjected to thermal and mechanical loads

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Abstract

The thermomechanical states in a class of functionally graded cylinders under extension, torsion, shearing, pressuring, and temperature changes are studied. Referred to the cylindrical coordinates, the material is cylindrically anisotropic. The only material symmetry is reflectional symmetry with respect to the cylindrical surfaces $r =$ constant. The material properties are considered to be radial dependent such that the conductivity coefficients, the thermal coefficients as well as the elastic constants depend in a specific manner on r. Exact solutions for the temperature distribution, thermoelastic deformations and stress fields are determined for inhomogeneous hollow and solid cylinders, with power law dependence of the moduli, subjected to an axial force and a torque at the ends and the surface loads that may vary circumferentially but not axially. In addition, exact solutions for thermoelastic responses of rotating cylinders are obtained within the context.

Keywords: Exact solution; Functionally graded cylinders; Inhomogeneous material; Thermoelasticity; Cylindrical anisotropy; Centrifugal force; Extension; Torsion

1. Introduction

Functionally graded materials (FGM) are composite materials intentionally designed so that they possess desirable properties for specific applications, especially for performance under thermal environment. On the macroscopic scale, FGMs are inhomogeneous with spatially varying material properties. Because of the combined effects of anisotropy and inhomogeneity, it is extremely difficult to obtain exact solutions for thermoelastic problems of FGM with anisotropic properties. Previous studies on the subject considered FG *isotropic* materials including those, for example, by Tanigawa (1995), Main and Spencer (1998), Horgan and Chan (1998, 1999), Zimmerman and Lutz (1999), Yang (2000), and Rooney and Ferrari (2001), where additional references can be found. Few work has been done for FG *anisotropic* materials. In studying the microstructural optimization of a FG layer, Nadeau and Ferrari (1999) presented a one dimensional thermal stress analysis of a transversely isotropic layer that is inhomogeneous in its

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thickness. Emphasis therein was on the optimization of the effective properties of the layer in terms of the microstructural parameters. Lekhnitskii (1981) in his monograph gave some analytical solutions for torsion of inhomogeneous circular cylinders, and plane deformation of a hollow cylinder under extension, bending and pressuring. The material was assumed to be *orthotropic* with a power variation of the elastic constants in the radial direction. The thermal effect was not considered.

In this paper we consider the thermomechanical states in a class of FG cylinders under extension, torsion, shearing, pressuring and temperature changes. The problem of inhomogeneous, solid or hollow circular cylinders subjected to thermomechanical loading is formulated in a state space setting in which the stress and the displacement are taken to be the state variables. Referred to the cylindrical coordinates (r, θ, z), the material is cylindrically anisotropic. The only material symmetry is reflectional symmetry with respect to the cylindrical surfaces $r =$ constant. Orthotropic, transversely isotropic and isotropic materials are included as special cases. Cylindrical anisotropy is not uncommon in the cylindrical body. It appears in carbon fiber (Dresselhaus et al., 1988; Christensen, 1994). The metallic forming process, such as extrusion or drawing, may result in cylindrically anisotropic products. Natural bamboo, tree trunk, and filamentary wound composite cylinders, in a broad sense, may be regarded as FG cylinders with cylindrically anisotropic material properties. We consider herein cylindrically anisotropic FG cylinders with material properties varying continuously across the cross-section such that the conductivity and the thermal coefficients as well as the elastic constants depend in a specific manner on r. The surfaces loads may vary circumferentially but not axially so that the thermoelastic field is independent of z. The end conditions require that the stress resultants reduce to an axial force and a torque. As such, the end effect is neglected. In addition, the centrifugal force due to rotating of the cylinder at a constant angular velocity is considered. The state space formalism makes it easy to determine the temperature distribution, thermoelastic deformation and stress field. In cases the material properties follow power law dependence on r, *exact* solutions are obtained. For arbitrary radially dependent material properties, the solution must be determined by numerical means. Alternatively, the FG cylinder may be modeled as a cylinder composed of fictitious coaxial layers and use the state space approach in conjunction with the transfer matrix for the laminated composite tubes (Tarn and Wang, 2001a,b) in obtaining the solution.

The analysis is conducted on the basis of uncoupled thermoelasticity (Boley and Weiner, 1960) by assuming that the thermomechanical loading varies slowly in time and the rate of entropy vanishes. Under this situation, the temperature field is constant in time and the thermoelastic state is stationary (Nowacki, 1975, 1986), the thermal and the mechanical problems are uncoupled. On determining the temperature field from heat conduction equations, it is regarded as a known function and is introduced in the thermoelastic equations to determine the thermal stresses. The first part of the paper presents the exact solutions for the steady state temperature distributions in inhomogeneous, hollow and solid cylinder under prescribed thermal boundary conditions. The second part presents the thermoelastic analysis of the cylinder subjected to thermomechanical loads and the centrifugal force. Exact solutions are obtained for the deformation and stress field in the cylinder under rotation as well as extension, torsion, shearing, pressuring and temperature changes.

2. Heat conduction

2.1. Basic equations

We consider the steady state temperature distribution in a cylindrically anisotropic circular cylinder under prescribed thermal boundary conditions. The material properties are assumed to be radially dependent but temperature independent. Referred to the circular cylindrical coordinates (r, θ, z), the Fourier law of heat conduction in an anisotropic solid may be expressed as (Özisik, 1993)

$$\begin{bmatrix} q_r \\ q_\theta \\ q_z \end{bmatrix} = -\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} T_{,r} \\ r^{-1}T_{,\theta} \\ T_{,z} \end{bmatrix}, \tag{1}$$

where q_r, q_θ, q_z are the heat flux, T denotes the temperature distribution in the body, a comma denotes partial differentiation with respect to the suffix variables, k_{ij} are the conductivity coefficients for the cylindrically anisotropic material, which are symmetric and limited by the requirement $k_{ii}k_{jj} - k_{ij}^2 > 0$ (no summation on i and j) for $i \neq j$. If the material is orthotropic with respect to the cylindrical coordinates, $k_{ij} = 0$ for $i \neq j$. If the material is *transversely isotropic* with respect to z-axis, $k_{11} = k_{22}$ in addition. When the material is isotropic, $k_{11} = k_{22} = k_{33}$. For the materials to be studied k_{ij} vary continuously in the radial direction so that $k_{ij} = k_{ij}(r)$.

The heat balance equation for steady-state heat conduction without heat generation is

$$r^{-1}(rq_r)_{,r} + r^{-1}q_{\theta,\theta} + q_{z,z} = 0. \tag{2}$$

When the body is subjected to thermal loads that are independent of z, the partial derivatives with respect to z in Eqs. (1) and (2) vanish. Formulating the problem in a state space setting by taking T, rq_r as the primary state variables and expressing rq_θ, rq_z in terms of them, we may cast Eqs. (1) and (2) into

$$r\frac{\partial}{\partial r}\begin{bmatrix} T \\ rq_r \end{bmatrix} = \begin{bmatrix} -k_{12}k_{11}^{-1}\partial_\theta & -k_{11}^{-1} \\ \widetilde{k}_{22}\partial_{\theta\theta} & -k_{12}k_{11}^{-1}\partial_\theta \end{bmatrix} \begin{bmatrix} T \\ rq_r \end{bmatrix}, \tag{3}$$

$$\begin{bmatrix} rq_\theta \\ rq_z \end{bmatrix} = \begin{bmatrix} -\widetilde{k}_{22}\partial_\theta & k_{12}k_{11}^{-1} \\ -\widetilde{k}_{23}\partial_\theta & k_{13}k_{11}^{-1} \end{bmatrix} \begin{bmatrix} T \\ rq_r \end{bmatrix}, \tag{4}$$

where $\widetilde{k}_{ij} = k_{ij} - k_{i1}k_{1j}k_{11}^{-1}$, ∂_θ and $\partial_{\theta\theta}$ denote the first and second order partial derivatives with respect to θ.

Three kinds of thermal boundary conditions may be considered: (1) prescribed surface temperature; (2) prescribed heat flux across the surface; (3) linear heat transfer on the surface. These conditions may be expressed as a linear combination of the temperature and the normal heat flux (Carslaw and Jaeger, 1959). For a solid cylinder with radius r_2 they can be written as

$$\left([h_1 \quad h_2/r] \begin{bmatrix} T \\ rq_r \end{bmatrix} \right)_{r=r_2} = f(\theta), \tag{5}$$

where $f(\theta)$ is a prescribed function of θ. The boundary conditions of the first and second kinds are obtained by setting h_2 and the heat transfer coefficient $h_1 = 0$, respectively.

For a hollow cylinder the boundary condition on the inner surface $r = r_1$ is

$$\left([h_1 \quad h_2/r] \begin{bmatrix} T \\ rq_r \end{bmatrix} \right)_{r=r_1} = g(\theta), \tag{6}$$

where $g(\theta)$ is a prescribed function.

We seek the solution to Eq. (3) in the complex form of the Fourier series

$$\begin{bmatrix} T \\ rq_r \end{bmatrix} = \sum_{n=-\infty}^{\infty} \begin{bmatrix} \widetilde{T}_n(r) \\ r\widetilde{q}_{rn}(r) \end{bmatrix} \mathrm{e}^{\mathrm{i}n\theta}, \tag{7}$$

where $\widetilde{T}_n(r)$ and $\widetilde{q}_{rn}(r)$ are complex functions of r to be determined.

Substituting Eq. (7) in Eqs. (3) and (4) gives

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} \widetilde{T}_n \\ r\widetilde{q}_{rn} \end{bmatrix} = \begin{bmatrix} -\mathrm{i}nk_{12}k_{11}^{-1} & -k_{11}^{-1} \\ -n^2\widetilde{k}_{22} & -\mathrm{i}nk_{12}k_{11}^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{T}_n \\ r\widetilde{q}_{rn} \end{bmatrix}, \tag{8}$$

$$\begin{bmatrix} rq_\theta \\ rq_z \end{bmatrix} = \sum_{n=-\infty}^{\infty} \begin{bmatrix} -\mathrm{i}n\widetilde{k}_{22} & k_{12}k_{11}^{-1} \\ -\mathrm{i}n\widetilde{k}_{23} & k_{13}k_{11}^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{T}_n \\ r\widetilde{q}_{rn} \end{bmatrix} \mathrm{e}^{\mathrm{i}n\theta}. \tag{9}$$

On expanding $f(\theta)$ and $g(\theta)$ in the complex Fourier series, Eqs. (5) and (6) become

$$\left([h_1 \quad h_2/r] \begin{bmatrix} \widetilde{T}_n \\ r\widetilde{q}_{rn} \end{bmatrix} \right)_{r=r_2} = A_n, \tag{10}$$

$$\left([h_1 \quad h_2/r] \begin{bmatrix} \widetilde{T}_n \\ r\widetilde{q}_{rn} \end{bmatrix} \right)_{r=r_1} = B_n, \tag{11}$$

where

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} f(\theta) \\ g(\theta) \end{bmatrix} \mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta, \quad (n = 0, \pm 1, \pm 2, \ldots).$$

In order to obtain the solution of Eq. (8) that satisfies Eq. (10) for a solid cylinder, or Eqs. (10) and (11) for a hollow cylinder, the dependence of k_{ij} on r must be specified. We consider that k_{ij} vary in proportional to some power of r such that

$$k_{ij}(r) = \kappa_{ij} r^m, \tag{12}$$

where κ_{ij} are given constants, m is a real number. For a solid cylinder m should be non-negative in order to avoid k_{ij} being unbounded at $r = 0$. Obviously, setting $m = 0$ reduces to the case of a homogeneous material.

This particular representation of the inhomogeneity make it possible to obtain exact solutions for the FGM in which the material properties are radially dependent. More general types of gradation can be treated using the modeling scheme to be described later.

2.2. Solution of the thermal field

With the conductivity coefficients specified by Eq. (12), Eq. (8) becomes

$$r\frac{\mathrm{d}}{\mathrm{d}r} \begin{bmatrix} \widetilde{T}_n \\ r\widetilde{q}_{rn} \end{bmatrix} = \begin{bmatrix} -\mathrm{i}n\kappa_{12}\kappa_{11}^{-1} & -r^{-m}\kappa_{11}^{-1} \\ -n^2\widetilde{\kappa}_{22} r^m & -\mathrm{i}n\kappa_{12}\kappa_{11}^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{T}_n \\ r\widetilde{q}_{rn} \end{bmatrix}, \tag{13}$$

where $\widetilde{\kappa}_{22} = \kappa_{22} - \kappa_{11}^{-1}\kappa_{12}^2$.

The solution to Eq. (13) takes the form

$$\begin{bmatrix} \widetilde{T}_n \\ r\widetilde{q}_{rn} \end{bmatrix} = \begin{bmatrix} c_1 r^\lambda \\ c_2 r^{\lambda+m} \end{bmatrix}, \tag{14}$$

where λ, c_1, c_2 are constants to be determined.

Substituting Eq. (14) into Eq. (13) gives a system of homogeneous algebraic equations. Non-trivial solution exists only if determinant of the coefficient matrix equal to zero. This yields

$$\lambda^2 + (2\mathrm{i}n\kappa_{12}\kappa_{11}^{-1} + m)\lambda + n(\mathrm{i}m\kappa_{12} - n\kappa_{22})\kappa_{11}^{-1} = 0, \tag{15}$$

from which λ is found to be

$$\begin{pmatrix} \lambda_{n1} \\ \lambda_{n2} \end{pmatrix} = 0.5\{-m \pm [m^2 + 4n^2(\kappa_{11}\kappa_{22} - \kappa_{12}^2)\kappa_{11}^{-2}]^{1/2}\} - \mathrm{i}n\kappa_{12}\kappa_{11}^{-1}, \tag{16}$$

where $n = 0, \pm 1, \pm 2, \ldots$, the part in the bracket is a real number because $\kappa_{11}\kappa_{22} - \kappa_{12}^2 > 0$.

It is easily seen that $\lambda_{01} = 0$, $\lambda_{02} = -m$ for the axisymmetric thermal field. Repeated roots $\lambda_{01} = \lambda_{02} = 0$ occur only when $m = 0$ for the axisymmetric thermal field in a *homogeneous* material, which has been treated in another publication (Tarn and Wang, 2000b). For the *inhomogeneous* materials under study the roots are always *distinct*. The roots for n and $-n$ are *complex conjugate*. When $n \neq 0$, $\mathrm{Re}(\lambda_{n1}) > 0$, $\mathrm{Re}(\lambda_{n1} + m) > 0$, $\mathrm{Re}(\lambda_{n2}) < 0$, $\mathrm{Re}(\lambda_{n2} + m) < 0$ for $m > 0$, where $\mathrm{Re}(\lambda)$ stands for the real part of λ. For orthotropic material λ_{n1} and λ_{n2} are *real* because $\kappa_{12} = 0$.

On substituting Eq. (16) into Eq. (13), there follows

$$\begin{bmatrix} \widetilde{T}_n(r) \\ r\widetilde{q}_{rn}(r) \end{bmatrix} = \begin{bmatrix} r^{\lambda_{n1}} & r^{\lambda_{n2}} \\ -(\mathrm{i}n\kappa_{12} + \kappa_{11}\lambda_{n1})r^{\lambda_{n1}+m} & -(\mathrm{i}n\kappa_{12} + \kappa_{11}\lambda_{n2})r^{\lambda_{n2}+m} \end{bmatrix} \begin{bmatrix} c'_{n1} \\ c'_{n2} \end{bmatrix}, \tag{17}$$

where c'_{n1} and c'_{n2} are constants to be determined from the boundary conditions.

For a hollow cylinder Eq. (17) must satisfy Eqs. (10) and (11). As a result, we have

$$\begin{bmatrix} c'_{n1} \\ c'_{n2} \end{bmatrix} = \begin{bmatrix} \phi(\lambda_{n1}, r_2) & \phi(\lambda_{n2}, r_2) \\ \phi(\lambda_{n1}, r_1) & \phi(\lambda_{n2}, r_1) \end{bmatrix}^{-1} \begin{bmatrix} A_n \\ B_n \end{bmatrix}, \tag{18}$$

where $\phi(\lambda, r) = h_1 r^{\lambda} - h_2(\mathrm{i}n\kappa_{12} + \kappa_{11}\lambda)r^{\lambda+m-1}$.

For a solid cylinder we must set $c'_{n2} = 0$ in Eq. (17) in order that the temperature and heat flux remain finite at $r = 0$. The constant c'_{n1} is determined by using Eq. (10) as

$$c'_{n1} = A_n/\phi(\lambda_{n1}, r_2). \tag{19}$$

Note that the solution for the solid cylinder cannot be obtained from that for the hollow cylinder by letting r_1 approach to zero. This will lead to unbounded temperature and heat flux at $r = 0$, which of course is untrue.

Having determined Eq. (17), we obtain the temperature distribution as

$$T = \sum_{n=-\infty}^{\infty} \widetilde{T}_n(r)\mathrm{e}^{\mathrm{i}n\theta} = \begin{cases} \sum_{n=-\infty}^{\infty} c'_{n1} r^{\lambda_{n1}} \mathrm{e}^{\mathrm{i}n\theta} & \text{for a solid cylinder}, \\ \sum_{n=-\infty}^{\infty} (c'_{n1} r^{\lambda_{n1}} + c'_{n2} r^{\lambda_{n2}}) \mathrm{e}^{\mathrm{i}n\theta} & \text{for a hollow cylinder}. \end{cases} \tag{20}$$

This concludes the solution of the steady-state heat conduction in FG cylinders. The temperature distribution will be incorporated in the subsequent thermoelastic analysis.

3. Thermoelastic analysis

3.1. State space formulation

We consider the FG cylinder of a cylindrically anisotropic material having at each point elastic symmetry with respect to the cylindrical surfaces r = constant. The thermoelastic stress–displacement relations are

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{13} & c_{23} & c_{33} & c_{34} & 0 & 0 \\ c_{14} & c_{24} & c_{34} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} u_{r,r} \\ r^{-1}(u_{\theta,\theta} + u_r) \\ u_{z,z} \\ u_{\theta,z} + r^{-1}u_{z,\theta} \\ u_{z,r} + u_{r,z} \\ r^{-1}u_{r,\theta} + u_{\theta,r} - r^{-1}u_\theta \end{bmatrix} - \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ 0 \\ 0 \end{bmatrix} T, \tag{21}$$

where the temperature distribution $T = T(r, \theta)$ is given by Eq. (20); $\sigma_r, \sigma_\theta, \ldots, \sigma_{r\theta}$ are the stress components; u_r, u_θ, u_z the displacement components; c_{ij} the 13 elastic constants and β_i the thermal coefficients of the material. The thermoelastic constants are radially dependent so that $c_{ij} = c_{ij}(r)$, $\beta_i = \beta_i(r)$.

If the material is orthotropic with respect to the cylindrical coordinates, $c_{14} = c_{24} = c_{34} = c_{56} = 0$, and $\beta_4 = 0$, the number of the independent elastic constants reduces to nine. If the material is transversely isotropic with respect to z-axis, further reduction is obtained with $c_{11} = c_{22}$, $c_{13} = c_{23}$, $c_{44} = c_{55}$, $c_{66} = (c_{11} - c_{12})/2$, and $\beta_1 = \beta_2$. When the material is isotropic, we have $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = c_{66} = \mu$, and $\beta_1 = \beta_2 = \beta_3$ in addition, where λ and μ are the Lamé constants.

When the cylinder is subjected to surface tractions that do not vary axially, the stress is independent of z. Following Tarn and Wang (2001a,b), taking u_r, u_θ, u_z and $r\sigma_r$, $r\sigma_{r\theta}$, $r\sigma_{rz}$ as the primary state variables, we may cast Eq. (21) and the equilibrium equations into a system of first order differential equations as follows:

$$r\frac{\partial}{\partial r}\begin{bmatrix} u_r \\ u_\theta \\ u_z \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz} \end{bmatrix} = \begin{bmatrix} -\hat{c}_{12} & d_{12} & d_{13} & c_{11}^{-1} & 0 & 0 \\ -\partial_\theta & 1 & 0 & 0 & s_{55} & s_{56} \\ -r\partial_z & 0 & 0 & 0 & s_{56} & s_{66} \\ Q_{22} & d_{42} & d_{43} & \hat{c}_{12} & -\partial_\theta & 0 \\ -Q_{22}\partial_\theta & d_{52} & d_{53} & -\hat{c}_{12}\partial_\theta & -1 & 0 \\ -Q_{24}\partial_\theta & d_{62} & d_{63} & -\hat{c}_{14}\partial_\theta & 0 & 0 \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \\ u_z \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz} \end{bmatrix} + r\begin{bmatrix} \beta_1 c_{11}^{-1} \\ 0 \\ 0 \\ -\widetilde{\beta}_2 \\ \widetilde{\beta}_2\partial_\theta \\ \widetilde{\beta}_4\partial_\theta \end{bmatrix} T - r\begin{bmatrix} 0 \\ 0 \\ 0 \\ R \\ \Theta \\ 0 \end{bmatrix}. \tag{22}$$

The in-surface stresses expressed in terms of the primary state variables are

$$\begin{bmatrix} r\sigma_\theta \\ r\sigma_z \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} Q_{22} & Q_{22}\partial_\theta + Q_{24}r\partial_z & Q_{24}\partial_\theta + Q_{23}r\partial_z \\ Q_{23} & Q_{23}\partial_\theta + Q_{34}r\partial_z & Q_{34}\partial_\theta + Q_{33}r\partial_z \\ Q_{24} & Q_{24}\partial_\theta + Q_{44}r\partial_z & Q_{44}\partial_\theta + Q_{34}r\partial_z \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} + \begin{bmatrix} \hat{c}_{12} \\ \hat{c}_{13} \\ \hat{c}_{14} \end{bmatrix} r\sigma_r - \begin{bmatrix} \widetilde{\beta}_2 \\ \widetilde{\beta}_3 \\ \widetilde{\beta}_4 \end{bmatrix} rT, \tag{23}$$

where R and Θ denote the body forces in the direction of r and θ, and

$$d_{12} = -(\hat{c}_{12}\partial_\theta + \hat{c}_{14}r\partial_z), \qquad d_{13} = -(\hat{c}_{14}\partial_\theta + \hat{c}_{13}r\partial_z), \qquad d_{42} = Q_{22}\partial_\theta + Q_{24}r\partial_z,$$

$$d_{43} = Q_{24}\partial_\theta + Q_{23}r\partial_z, \qquad d_{52} = -\partial_\theta(Q_{22}\partial_\theta + Q_{24}r\partial_z), \qquad d_{53} = -\partial_\theta(Q_{24}\partial_\theta + Q_{23}r\partial_z),$$

$$d_{62} = -\partial_\theta(Q_{24}\partial_\theta + Q_{44}r\partial_z), \qquad d_{63} = -\partial_\theta(Q_{44}\partial_\theta + Q_{34}r\partial_z),$$

$$\hat{c}_{ij} = c_{ij}/c_{11}, \qquad Q_{ij} = c_{ij} - c_{1i}c_{1j}/c_{11}, \qquad \widetilde{\beta}_i = \beta_i - \beta_1 c_{1i}/c_{11},$$

$$\begin{bmatrix} s_{55} & s_{56} \\ s_{56} & s_{66} \end{bmatrix} = \frac{1}{c_{55}c_{66} - c_{56}^2}\begin{bmatrix} c_{66} & -c_{56} \\ -c_{56} & c_{55} \end{bmatrix}.$$

When surface tractions are prescribed on $r = r_2$ of a solid cylinder, the boundary condition is

$$[r\sigma_r \quad r\sigma_{r\theta} \quad r\sigma_{rz}]_{r=r_2} = [r_2 p_2(\theta) \quad 0 \quad 0], \tag{24}$$

where $p_2(\theta)$ is the prescribed traction in the r direction, which may vary circumferentially. The condition that the stresses do not vary along the z-axis does not allow for non-zero tractions in θ and z directions for a solid cylinder.

For a hollow cylinder the boundary conditions are

$$[r\sigma_r \quad r\sigma_{r\theta} \quad r\sigma_{rz}]_{r=r_1} = [r_1 p_1(\theta) \quad r_1\tau_1(\theta) \quad r_1 s_1(\theta)], \tag{25}$$

$$[r\sigma_r \quad r\sigma_{r\theta} \quad r\sigma_{rz}]_{r=r_2} = [r_2 p_2(\theta) \quad r_2\tau_2(\theta) \quad r_2 s_2(\theta)], \tag{26}$$

where p_1, τ_1, s_1 and p_2, τ_2, s_2 are the tractions prescribed on the inner and outer surfaces, respectively. The tractions in the θ and z directions are admissible provided that they satisfy the conditions

$$r_1^2\int_0^{2\pi}\tau_1(\theta)\,\mathrm{d}\theta = r_2^2\int_0^{2\pi}\tau_2(\theta)\,\mathrm{d}\theta, \qquad r_1\int_0^{2\pi}s_1(\theta)\,\mathrm{d}\theta = r_2\int_0^{2\pi}s_2(\theta)\,\mathrm{d}\theta, \tag{27}$$

in order to maintain static equilibrium and produce the stress independent of z.

The end conditions require that the stress resultants over the cross-section reduce to an axial force P_z, a torque M_t, and bi-axial bending moments M_1, M_2, such that

$$\int_0^{2\pi}\int_{r_1}^{r_2}(r\sigma_z)\,\mathrm{d}r\,\mathrm{d}\theta = P_z, \tag{28}$$

$$\int_0^{2\pi}\int_{r_1}^{r_2}(r\sigma_{\theta z})r\,\mathrm{d}r\,\mathrm{d}\theta = M_t, \tag{29}$$

$$\int_0^{2\pi}\int_{r_1}^{r_2}(r\sigma_z)r\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = M_1, \tag{30}$$

$$\int_0^{2\pi}\int_{r_1}^{r_2}(r\sigma_z)r\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta = M_2. \tag{31}$$

In addition, the resultant shear forces must vanish on the ends. The conditions are satisfied identically when the stress is independent of z.

When the stress is independent of z, the general expressions for the displacement field are (Lekhnitskii, 1981)

$$u_r = u(r,\theta) - z^2(A\cos\theta + B\sin\theta)/2 + u_0, \tag{32}$$

$$u_\theta = v(r,\theta) + z^2(A\sin\theta - B\cos\theta)/2 + \vartheta rz + v_0, \tag{33}$$

$$u_z = w(r,\theta) + z(Ar\cos\theta + Br\sin\theta + \varepsilon) + w_0, \tag{34}$$

where u, v, w are unknown functions of r and θ; u_0, v_0, w_0 are associated with the rigid body displacements; the constants ε is a uniform extension, ϑ is the twisting angle per unit length along z-axis, A and B are associated with bending of the cylinder.

On substituting Eqs. (32) and (33) in Eqs. (22) and (23) and using

$$\sin\theta = \mathrm{i}(\mathrm{e}^{-\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta})/2, \qquad \cos\theta = (\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta})/2,$$

these equations become

$$r\frac{\partial}{\partial r}\begin{bmatrix} u \\ v \\ w \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz}\end{bmatrix} = \begin{bmatrix} -\hat{c}_{12} & -\hat{c}_{12}\partial_\theta & -\hat{c}_{14}\partial_\theta & c_{11}^{-1} & 0 & 0 \\ -\partial_\theta & 1 & 0 & 0 & s_{55} & s_{56} \\ 0 & 0 & 0 & 0 & s_{56} & s_{66} \\ Q_{22} & Q_{22}\partial_\theta & Q_{24}\partial_\theta & \hat{c}_{12} & -\partial_\theta & 0 \\ -Q_{22}\partial_\theta & -Q_{22}\partial_{\theta\theta} & -Q_{24}\partial_{\theta\theta} & -\hat{c}_{12}\partial_\theta & -1 & 0 \\ -Q_{24}\partial_\theta & -Q_{24}\partial_{\theta\theta} & -Q_{44}\partial_{\theta\theta} & -\hat{c}_{14}\partial_\theta & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz}\end{bmatrix} - r\begin{bmatrix} 0 \\ 0 \\ 0 \\ R \\ \Theta \\ 0\end{bmatrix}$$
$$+ Dr^2\mathrm{e}^{\mathrm{i}\theta}\begin{bmatrix} -\hat{c}_{13} \\ 0 \\ 0 \\ Q_{23} \\ -\mathrm{i}Q_{23} \\ -\mathrm{i}Q_{34}\end{bmatrix} + \overline{D}r^2\mathrm{e}^{-\mathrm{i}\theta}\begin{bmatrix} -\hat{c}_{13} \\ 0 \\ 0 \\ Q_{23} \\ \mathrm{i}Q_{23} \\ \mathrm{i}Q_{34}\end{bmatrix} + \varepsilon r\begin{bmatrix} -\hat{c}_{13} \\ 0 \\ 0 \\ Q_{23} \\ 0 \\ 0\end{bmatrix} + \vartheta r^2\begin{bmatrix} 0 \\ 0 \\ 0 \\ Q_{24} \\ 0 \\ 0\end{bmatrix} + r\begin{bmatrix} \beta_1 c_{11}^{-1} \\ 0 \\ 0 \\ -\widetilde{\beta}_2 \\ \widetilde{\beta}_2\partial_\theta \\ \widetilde{\beta}_4\partial_\theta\end{bmatrix}T, \tag{35}$$

$$\begin{bmatrix} r\sigma_\theta \\ r\sigma_z \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} Q_{22} & Q_{22}\partial_\theta & Q_{24}\partial_\theta \\ Q_{23} & Q_{23}\partial_\theta & Q_{34}\partial_\theta \\ Q_{24} & Q_{24}\partial_\theta & Q_{44}\partial_\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} \hat{c}_{12} \\ \hat{c}_{13} \\ \hat{c}_{14} \end{bmatrix} r\sigma_r - \begin{bmatrix} \widetilde{\beta}_2 \\ \widetilde{\beta}_3 \\ \widetilde{\beta}_4 \end{bmatrix} rT + (Dr^2 \mathrm{e}^{\mathrm{i}\theta} + \overline{D}r^2 \mathrm{e}^{-\mathrm{i}\theta} + \varepsilon r) \begin{bmatrix} Q_{23} \\ Q_{33} \\ Q_{34} \end{bmatrix} + \vartheta r^2 \begin{bmatrix} Q_{24} \\ Q_{34} \\ Q_{44} \end{bmatrix}, \tag{36}$$

where $D = (A - \mathrm{i}B)/2$, $\overline{D} = (A + \mathrm{i}B)/2$. Henceforth the over-bar denotes complex conjugate.

3.2. General solution

We seek the solution to Eq. (35) in the form of complex Fourier series:

$$[\,u \quad v \quad w \quad r\sigma_r \quad r\sigma_{r\theta} \quad r\sigma_{rz}\,] = \sum_{n=-\infty}^{\infty} [\,U_n \quad V_n \quad W_n \quad X_n \quad Y_n \quad Z_n\,]\mathrm{e}^{\mathrm{i}n\theta}, \tag{37}$$

where $U_n, V_n, \ldots, Z_n$ are unknown complex functions of r.

Substituting Eqs. (20) and (37) in Eq. (35) enables us to decompose the equation into sets of ordinary differential equations that can be solved explicitly when the dependence of c_{ij} and β_i on r are specified. Let us consider the power law variation of the thermoelastic constants c_{ij} and β_i such that

$$c_{ij}(r) = a_{ij} r^k, \qquad \beta_i(r) = b_i r^k, \tag{38}$$

where a_{ij} and b_i are given constants, k is a real number. In order that the thermoelastic constants for a solid cylinder are finite, k must be non-negative. The special case of a homogeneous material is obtained by letting $k = 0$. The power law variation was often assumed in stress analysis of FG cylinders (Lekhnitskii, 1981; Horgan and Chan, 1999; Yang, 2000). It is the simplest way of representing the thermoelastic FG cylinder that affords exact solutions.

Substituting Eqs. (37) and (38) in Eq. (35) results in the following sets of matrix differential equations.

(1) For $n = 0$,

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} \mathbf{U}_0 \\ \mathbf{S}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{01} & r^{-k}\mathbf{N}_{02} \\ r^k\mathbf{N}_{03} & -\mathbf{N}_{01}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{U}_0 \\ \mathbf{S}_0 \end{bmatrix} + \varepsilon\begin{bmatrix} r\boldsymbol{\phi}_{u0} \\ r^{k+1}\boldsymbol{\phi}_{s0} \end{bmatrix} + \vartheta\begin{bmatrix} \mathbf{0} \\ r^{k+2}\boldsymbol{\varphi}_{s0} \end{bmatrix} + \widetilde{T}_0\begin{bmatrix} r\boldsymbol{\psi}_{u0} \\ r^{k+1}\boldsymbol{\psi}_{s0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ r\mathbf{R} \end{bmatrix}, \tag{39}$$

where $\mathbf{U}_0 = [\,U_0 \quad V_0 \quad W_0\,]^{\mathrm{T}}$, $\mathbf{S}_0 = [\,X_0 \quad Y_0 \quad Z_0\,]^{\mathrm{T}}$. Other notations used in Eq. (39) and in the following equations are given in Appendix A for clarity.

Eq. (39) is the governing equation for the axisymmetric deformation and stresses in the cylinder subjected to thermomechanical loads that are independent of θ.

(2) For $n = \pm 1$,

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} \mathbf{U}_1 \\ \mathbf{S}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{11} & r^{-k}\mathbf{N}_{12} \\ r^k\mathbf{N}_{13} & -\overline{\mathbf{N}}_{11}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{U}_1 \\ \mathbf{S}_1 \end{bmatrix} + D\begin{bmatrix} r^2\boldsymbol{\phi}_{u1} \\ r^{k+2}\boldsymbol{\phi}_{s1} \end{bmatrix} + \widetilde{T}_1\begin{bmatrix} r\boldsymbol{\psi}_{u1} \\ r^{k+1}\boldsymbol{\psi}_{s1} \end{bmatrix}, \tag{40}$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} \mathbf{U}_{-1} \\ \mathbf{S}_{-1} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{N}}_{11} & r^{-k}\overline{\mathbf{N}}_{12} \\ r^k\overline{\mathbf{N}}_{13} & -\mathbf{N}_{11}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{U}_{-1} \\ \mathbf{S}_{-1} \end{bmatrix} + \overline{D}\begin{bmatrix} r^2\overline{\boldsymbol{\phi}}_{u1} \\ r^{k+2}\overline{\boldsymbol{\phi}}_{s1} \end{bmatrix} + \widetilde{T}_{-1}\begin{bmatrix} r\overline{\boldsymbol{\psi}}_{u1} \\ r^{k+1}\overline{\boldsymbol{\psi}}_{s1} \end{bmatrix}, \tag{41}$$

where

$$\mathbf{U}_1 = [\,U_1 \quad V_1 \quad W_1\,]^{\mathrm{T}}, \qquad \mathbf{S}_1 = [\,X_1 \quad Y_1 \quad Z_1\,]^{\mathrm{T}},$$

$$\mathbf{U}_{-1} = [\,U_{-1} \quad V_{-1} \quad W_{-1}\,]^{\mathrm{T}}, \qquad \mathbf{S}_{-1} = [\,X_{-1} \quad Y_{-1} \quad Z_{-1}\,]^{\mathrm{T}}.$$

Eqs. (40) and (41) are complex conjugate, so are their solutions.

(3) For $n = \pm 2, \pm 3, \ldots,$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix}\mathbf{U}_n\\ \mathbf{S}_n\end{bmatrix} = \begin{bmatrix}\mathbf{N}_{n1} & r^{-k}\mathbf{N}_{n2}\\ r^k\mathbf{N}_{n3} & -\overline{\mathbf{N}}_{n1}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{U}_n\\ \mathbf{S}_n\end{bmatrix} + \widetilde{T}_n\begin{bmatrix}r\boldsymbol{\psi}_{un}\\ r^{k+1}\boldsymbol{\psi}_{sn}\end{bmatrix}, \tag{42}$$

where

$$\mathbf{U}_n = [U_n \quad V_n \quad W_n]^{\mathrm{T}}, \qquad \mathbf{S}_n = [X_n \quad Y_n \quad Z_n]^{\mathrm{T}}.$$

Eqs. (39)–(42) are of the same form except for the non-homogeneous terms. The complete solution consists of the homogeneous solution and the particular solution. The homogeneous solution takes the form

$$\begin{bmatrix}\mathbf{U}_n\\ \mathbf{S}_n\end{bmatrix}_h = \begin{bmatrix}\widetilde{\mathbf{U}}_n r^{\mu}\\ \widetilde{\mathbf{S}}_n r^{\mu+k}\end{bmatrix}, \tag{43}$$

for $n = 0, \pm 1, \pm 2, \ldots,$ where μ is an unknown constant and $\widetilde{\mathbf{U}}_n$, $\widetilde{\mathbf{S}}_n$ are constant vectors to be determined.

Substituting Eq. (43) in Eqs. (39)–(42) leads to an eigenvalue problem

$$\begin{bmatrix}\mathbf{N}_{n1} & \mathbf{N}_{n2}\\ \mathbf{N}_{n3} & -\overline{\mathbf{N}}_{n1}^{\mathrm{T}} - k\mathbf{I}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{U}}_n\\ \widetilde{\mathbf{S}}_n\end{bmatrix} = \mu\begin{bmatrix}\widetilde{\mathbf{U}}_n\\ \widetilde{\mathbf{S}}_n\end{bmatrix}, \tag{44}$$

in which μ is the eigenvalue and $\begin{bmatrix}\widetilde{\mathbf{U}}_n & \widetilde{\mathbf{S}}_n\end{bmatrix}^{\mathrm{T}}$ the eigenvector.

Non-trivial solution of Eq. (44) exists provided that

$$\begin{vmatrix}\mathbf{N}_{n1} - \mu\mathbf{I} & \mathbf{N}_{n2}\\ \mathbf{N}_{n3} & -\overline{\mathbf{N}}_{n1}^{\mathrm{T}} - k\mathbf{I} - \mu\mathbf{I}\end{vmatrix} = 0, \tag{45}$$

from which six eigenvalues μ_{ni} can be determined for each value of n. To each eigenvalue μ_{ni} there corresponds an eigenvector $\begin{bmatrix}\widetilde{\mathbf{U}}_n & \widetilde{\mathbf{S}}_n\end{bmatrix}_j^{\mathrm{T}}$. The linear combination of the eigensolutions is again the homogeneous solution of Eqs. (39)–(42).

The particular solution for Eqs. (40) and (41) are complex conjugate. The one for Eq. (40) takes the form

$$\begin{bmatrix}\mathbf{U}_1\\ \mathbf{S}_1\end{bmatrix}_p = \begin{bmatrix}c_1 r\\ c_2 r^{k+1}\end{bmatrix} + \sum_{j=1}^{2}\begin{bmatrix}a_{1j}r^{\lambda_{1j}+1}\\ b_{1j}r^{\lambda_{1j}+k+1}\end{bmatrix}, \tag{46}$$

where λ_{1j} are given by Eq. (16), and c_1, c_2, a_{1j}, b_{1j} are constants to be determined.

Substituting Eq. (46) in Eq. (40) gives us

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix} = D\begin{bmatrix}2\mathbf{I}_3 - \mathbf{N}_{11} & -\mathbf{N}_{12}\\ -\mathbf{N}_{13} & (k+2)\mathbf{I}_3 + \overline{\mathbf{N}}_{11}^{\mathrm{T}}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{\phi}_{u1}\\ \boldsymbol{\phi}_{s1}\end{bmatrix}, \tag{47}$$

$$\begin{bmatrix}a_{1j}\\ b_{1j}\end{bmatrix} = \begin{bmatrix}(\lambda_{1j}+1)\mathbf{I}_3 - \mathbf{N}_{11} & -\mathbf{N}_{12}\\ -\mathbf{N}_{13} & (\lambda_{1j}+k+1)\mathbf{I}_3 + \overline{\mathbf{N}}_{11}^{\mathrm{T}}\end{bmatrix}^{-1}\begin{bmatrix}c'_{1j}\boldsymbol{\psi}_{u1}\\ c'_{1j}\boldsymbol{\psi}_{s1}\end{bmatrix}. \tag{48}$$

Similarly, the particular solution of Eq. (42) is found to be

$$\begin{bmatrix}\mathbf{U}_n\\ \mathbf{S}_n\end{bmatrix}_p = \sum_{j=1}^{2}\begin{bmatrix}a_{nj}r^{\lambda_{nj}+1}\\ b_{nj}r^{\lambda_{nj}+k+1}\end{bmatrix}, \tag{49}$$

where

$$\begin{bmatrix} a_{nj} \\ b_{nj} \end{bmatrix} = \begin{bmatrix} (\lambda_{nj}+1)\mathbf{I}_3 - \mathbf{N}_{n1} & -\mathbf{N}_{n2} \\ -\mathbf{N}_{n3} & (\lambda_{nj}+k+1)\mathbf{I}_3 + \overline{\mathbf{N}}_{n1}^{\mathrm{T}} \end{bmatrix}^{-1} \begin{bmatrix} c'_{nj}\boldsymbol{\psi}_{un} \\ c'_{nj}\boldsymbol{\psi}_{sn} \end{bmatrix}.$$

In presenting the particular solution we did not include the remote possibility that the eigenvalue μ may coincide with the values of $\lambda_{nj}+1$. If it does occur, the form of the particular solution needs to be modified by multiplying the associated terms by $\log r$. This can be easily done.

The complete solution of Eqs. (39)–(42) consists of a linear combination of Eq. (43) plus the particular solution

$$\begin{bmatrix} \mathbf{U}_n \\ \mathbf{S}_n \end{bmatrix} = \sum_{j=1}^{6} c_j \begin{bmatrix} \widetilde{\mathbf{U}}_n r^{\mu_j} \\ \widetilde{\mathbf{S}}_n r^{\mu_j+k} \end{bmatrix} + \begin{bmatrix} \mathbf{U}_n \\ \mathbf{S}_n \end{bmatrix}_p. \tag{50}$$

The constants of the linear combination c_j are determined by requiring Eq. (50) satisfy the boundary conditions. To this end, we expand the surface tractions in the complex Fourier series to reduce Eqs. (24)–(26) to

$$(\mathbf{S}_n)_{r=r_2} = [X_n \quad Y_n \quad Z_n]_{r=r_2} = [r\widetilde{p}_{2n} \quad 0 \quad 0]_{r=r_2} \tag{51}$$

for a solid cylinder, and

$$(\mathbf{S}_n)_{r=r_1} = [X_n \quad Y_n \quad Z_n]_{r=r_1} = [r\widetilde{p}_{1n} \quad r\widetilde{\tau}_{1n} \quad r\widetilde{s}_{1n}]_{r=r_1}, \tag{52}$$

$$(\mathbf{S}_n)_{r=r_2} = [X_n \quad Y_n \quad Z_n]_{r=r_2} = [r\widetilde{p}_{2n} \quad r\widetilde{\tau}_{2n} \quad r\widetilde{s}_{2n}]_{r=r_2}, \tag{53}$$

for a hollow cylinder, where $n = 0, \pm 1, \pm 2, \ldots,$ and

$$[\widetilde{p}_{2n} \quad \widetilde{\tau}_{2n} \quad \widetilde{s}_{2n} \quad \widetilde{p}_{1n} \quad \widetilde{\tau}_{1n} \quad \widetilde{s}_{1n}] = \frac{1}{2\pi}\int_0^{2\pi} [p_2 \quad \tau_2 \quad s_2 \quad p_1 \quad \tau_1 \quad s_1] \mathrm{e}^{-\mathrm{i}n\theta}\, \mathrm{d}\theta. \tag{54}$$

With Eqs. (52) and (53), the six constants c_j in Eq. (50) are uniquely determined for a hollow cylinder. For a solid cylinder Eq. (52) is replaced by the condition that the displacements and stresses at the center must be finite. The condition demands that the terms with negative power of r be excluded from the solution.

The in-surface stresses are obtained by substituting Eqs. (37) and (50) in Eq. (36) as

$$\begin{aligned} \begin{bmatrix} \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \end{bmatrix} = \sum_{n=-\infty}^{\infty} \left(r^{k-1} \begin{bmatrix} \widetilde{Q}_{22} & \mathrm{i}n\widetilde{Q}_{22} & \mathrm{i}n\widetilde{Q}_{24} \\ \widetilde{Q}_{23} & \mathrm{i}n\widetilde{Q}_{23} & \mathrm{i}n\widetilde{Q}_{34} \\ \widetilde{Q}_{24} & \mathrm{i}n\widetilde{Q}_{24} & \mathrm{i}n\widetilde{Q}_{44} \end{bmatrix} \begin{bmatrix} U_n \\ V_n \\ W_n \end{bmatrix} + r^{-1} \begin{bmatrix} \widehat{a}_{12} \\ \widehat{a}_{13} \\ \widehat{a}_{14} \end{bmatrix} X_n - r^k \widetilde{T}_n \begin{bmatrix} \widetilde{b}_2 \\ \widetilde{b}_3 \\ \widetilde{b}_4 \end{bmatrix} \right) \mathrm{e}^{\mathrm{i}n\theta} \\ + r^k (Dr\mathrm{e}^{\mathrm{i}\theta} + \overline{D}r\mathrm{e}^{-\mathrm{i}\theta} + \varepsilon) \begin{bmatrix} \widetilde{Q}_{23} \\ \widetilde{Q}_{33} \\ \widetilde{Q}_{34} \end{bmatrix} + \vartheta r^{k+1} \begin{bmatrix} \widetilde{Q}_{24} \\ \widetilde{Q}_{34} \\ \widetilde{Q}_{44} \end{bmatrix}. \end{aligned} \tag{55}$$

4. Exact solutions for axisymmetric response

When the applied load is independent of θ, the deformation and stresses in the cylinder are axisymmetric. The thermomechanical loading that gives rise to the axisymmetric response includes internal and external pressure, uniform surface shears, an axial force, a torque, axisymmetric body forces and the axisymmetric temperature distribution given by

$$\widetilde{T}_0 = \begin{cases} c'_{01} & \text{for a solid cylinder,} \\ c'_{01} + c'_{02} r^{-m} & \text{for a hollow cylinder,} \end{cases} \tag{56}$$

which is the $n = 0$ term of Eq. (20), where the constants c'_{01} and c'_{02} have been determined in Section 2.2.

4.1. Hollow cylinder

The governing equation (39) for the axisymmetric response may be uncoupled into

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = \begin{bmatrix} -a_{12}a_{11}^{-1} & r^{-k}a_{11}^{-1} \\ r^k\widetilde{Q}_{22} & a_{12}a_{11}^{-1} \end{bmatrix}\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} + \varepsilon\begin{bmatrix} -a_{13}a_{11}^{-1}r \\ \widetilde{Q}_{23}r^{k+1} \end{bmatrix} + \vartheta\begin{bmatrix} 0 \\ \widetilde{Q}_{24}r^{k+2} \end{bmatrix} - (c'_{01} + c'_{02}r^{-m})\begin{bmatrix} -b_1a_{11}^{-1}r \\ \widetilde{b}_2r^{k+1} \end{bmatrix}, \tag{57}$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}Y_0 = -Y_0, \qquad r\frac{\mathrm{d}}{\mathrm{d}r}Z_0 = 0, \tag{58}$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}V_0 = V_0 + (\widetilde{s}_{55}Y_0 + \widetilde{s}_{56}Z_0)r^{-k}, \qquad r\frac{\mathrm{d}}{\mathrm{d}r}W_0 = (\widetilde{s}_{56}Y_0 + \widetilde{s}_{66}Z_0)r^{-k}, \tag{59}$$

in Eq. (57) the body force term has been excluded.

For axisymmetric response the tractions on $r = r_1$ and $r = r_2$ must be uniform such that

$$(X_0)_{r=r_1} = r_1p_1, \qquad (X_0)_{r=r_2} = r_2p_2; \tag{60}$$

$$(Y_0)_{r=r_1} = r_1\tau_1, \qquad (Y_0)_{r=r_2} = r_2\tau_2; \tag{61}$$

$$(Z_0)_{r=r_1} = r_1s_1, \qquad (Z_0)_{r=r_2} = r_2s_2, \tag{62}$$

where p_1, p_2; τ_1, τ_2; s_1, s_2 are the radial, circumferential, and axial components of the uniform tractions. For static equilibrium it requires $\tau_1r_1^2 = \tau_2r_2^2$ and $s_1r_1 = s_2r_2$.

The solution to Eq. (58) with the boundary conditions (61) and (62) are

$$Y_0 = \tau_1r_1^2/r = \tau_2r_2^2/r, \qquad Z_0 = s_1r_1 = s_2r_2, \tag{63}$$

so that

$$\sigma_{r\theta} = \tau_1r_1^2/r^2 = \tau_2r_2^2/r^2, \qquad \sigma_{rz} = s_1r_1/r = s_2r_2/r. \tag{64}$$

The results indicate that uniform shearing produces *pure shears* in the hollow cylinder of an inhomogeneous, monoclinic cylindrically anisotropic material. Note that the stress field is independent of the material properties.

Solving Eq. (59) with Eq. (63), we obtain

$$\begin{bmatrix} V_0 \\ W_0 \end{bmatrix} = \begin{bmatrix} cr \\ 0 \end{bmatrix} - \begin{bmatrix} \widetilde{s}_{55}/(k+2) & \widetilde{s}_{56}/(k+1) \\ \widetilde{s}_{56}/(k+1) & \widetilde{s}_{66}/k \end{bmatrix}\begin{bmatrix} \tau_2r_2^2r^{-(k+1)} \\ s_2r_2r^{-k} \end{bmatrix}, \tag{65}$$

where cr is a rigid body displacement.

The displacements u_θ and u_z are obtained by substituting Eq. (65) in Eqs. (33) and (34), giving

$$\begin{bmatrix} u_\theta \\ u_z \end{bmatrix} = \begin{bmatrix} \vartheta rz \\ \varepsilon z \end{bmatrix} - \begin{bmatrix} \widetilde{s}_{55}/(k+2) & \widetilde{s}_{56}/(k+1) \\ \widetilde{s}_{56}/(k+1) & \widetilde{s}_{66}/k \end{bmatrix}\begin{bmatrix} \tau_2r_2^2r^{-(k+1)} \\ s_2r_2r^{-k} \end{bmatrix}, \tag{66}$$

where the rigid body displacements have been excluded. The term $u_\theta = \vartheta rz$ represents torsion of the cylinder with the cross-section undistorted but rotating about z-axis. The term $u_z = \varepsilon z$ represents warping of the cross-section. The displacements due to the transverse shear and longitudinal shear are coupled through the elastic constant c_{56}. In the cases of orthotropy, transverse isotropy, or isotropy, $c_{56} = 0$, they are not coupled.

The complete solution of Eq. (57) consists of the homogeneous solution and the particular solution. The homogeneous solution takes the form

$$[\, U_0 \quad X_0 \,]_h = [\, \widetilde{U}_0 r^{\mu} \quad \widetilde{X}_0 r^{\mu+k} \,]. \tag{67}$$

Substituting Eq. (67) in the homogeneous equation of Eq. (57) yields a system of homogeneous algebraic equations. Non-trivial solution exists if the determinant of the coefficient matrix equals to zero, from which two distinct roots

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0.5\{-k \pm [k^2 + 4(a_{22} - ka_{12})a_{11}^{-1}]^{1/2}\} \tag{68}$$

are obtained. Mathematically, when $k^2 + 4(a_{22} - ka_{12})a_{11}^{-1} = 0$, the roots are repeated. But in view of $a_{12}^2 < a_{11}a_{22}$ (because the strain energy density function is positive definite), it is physically impossible since this requires k to be a complex number.

There follows

$$\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}_h = \begin{bmatrix} r^{\mu_1} & r^{\mu_2} \\ \widetilde{a}_1 r^{\mu_1+k} & \widetilde{a}_2 r^{\mu_2+k} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{69}$$

where $\widetilde{a}_1 = a_{12} + \mu_1 a_{11}$, $\widetilde{a}_2 = a_{12} + \mu_2 a_{11}$, c_1 and c_2 are undetermined constants of the linear combination.

The particular solution of Eq. (57) is

$$\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}_p = \varepsilon \begin{bmatrix} a_1 r \\ a_2 r^{k+1} \end{bmatrix} + \vartheta \begin{bmatrix} a_3 r^2 \\ a_4 r^{k+2} \end{bmatrix} + \begin{bmatrix} a_5 r \\ a_6 r^{k+1} \end{bmatrix} + \begin{bmatrix} a_7 r^{1-m} \\ a_8 r^{k+1-m} \end{bmatrix}, \tag{70}$$

where

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{k(a_{11} + a_{12}) + a_{11} - a_{22}} \begin{bmatrix} a_{12}a_{11}^{-1} - \widetilde{Q}_{23} - k - 1 \\ a_{13}\widetilde{Q}_{22} + (a_{11} + a_{12})\widetilde{Q}_{23} \end{bmatrix},$$

$$\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \frac{\widetilde{Q}_{24}}{k(a_{11} + 2a_{12}) + 4a_{11} - a_{22}} \begin{bmatrix} 1 \\ 2a_{11} + a_{12} \end{bmatrix},$$

$$\begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \frac{c'_{01}}{k(a_{11} + a_{12}) + a_{11} - a_{22}} \begin{bmatrix} b_1(k + 1 - a_{12}a_{11}^{-1}) + \widetilde{b}_2 \\ -b_1\widetilde{Q}_{22} - (a_{11} + a_{12})\widetilde{b}_2 \end{bmatrix},$$

$$\begin{bmatrix} a_7 \\ a_8 \end{bmatrix} = \frac{c'_{02}}{(k + 1 - m)(1 - m)a_{11} + ka_{12} - a_{22}} \begin{bmatrix} b_1(k + 1 - m - a_{12}a_{11}^{-1}) + \widetilde{b}_2 \\ b_1\widetilde{Q}_{22} + [(1 - m)a_{11} + a_{12}]\widetilde{b}_2 \end{bmatrix}.$$

On combining Eqs. (69) and (70), and imposing the boundary conditions (60) to determine the constants c_1 and c_2, we obtain

$$\begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} = \begin{bmatrix} r^{\mu_1} & r^{\mu_2} \\ \widetilde{a}_1 r^{\mu_1+k-1} & \widetilde{a}_2 r^{\mu_2+k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \varepsilon \begin{bmatrix} a_1 r \\ a_2 r^{k} \end{bmatrix} + \vartheta \begin{bmatrix} a_3 r^2 \\ a_4 r^{k+1} \end{bmatrix} + \begin{bmatrix} a_5 r \\ a_6 r^{k} \end{bmatrix} + \begin{bmatrix} a_7 r^{1-m} \\ a_8 r^{k-m} \end{bmatrix}, \tag{71}$$

where

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \widetilde{a}_1 r_1^{\mu_1+k-1} & \widetilde{a}_2 r_1^{\mu_2+k-1} \\ \widetilde{a}_1 r_2^{\mu_1+k-1} & \widetilde{a}_2 r_2^{\mu_2+k-1} \end{bmatrix}^{-1} \begin{bmatrix} p_1 - \varepsilon a_2 r_1^k - \vartheta a_4 r_1^{k+1} - a_6 r_1^k - a_8 r_1^{k-m} \\ p_2 - \varepsilon a_2 r_2^k - \vartheta a_4 r_2^{k+1} - a_6 r_2^k - a_8 r_2^{k-m} \end{bmatrix}.$$

The in-surface stresses are given by the $n = 0$ term of Eq. (55) in which Eq. (71) is substituted,

$$\begin{bmatrix} \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} \gamma_2(\mu_1) r^{\mu_1+k-1} & \gamma_2(\mu_2) r^{\mu_2+k-1} \\ \gamma_3(\mu_1) r^{\mu_1+k-1} & \gamma_3(\mu_2) r^{\mu_2+k-1} \\ \gamma_4(\mu_1) r^{\mu_1+k-1} & \gamma_4(\mu_2) r^{\mu_2+k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \varepsilon \begin{bmatrix} \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} r^k + \vartheta \begin{bmatrix} \widetilde{\eta}_2 \\ \widetilde{\eta}_3 \\ \widetilde{\eta}_4 \end{bmatrix} r^{k+1} + \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} r^k + \begin{bmatrix} \widetilde{\lambda}_2 \\ \widetilde{\lambda}_3 \\ \widetilde{\lambda}_4 \end{bmatrix} r^{k-m}, \tag{72}$$

where $\gamma_j(\mu) = a_{2j} + \mu a_{1j}$, $j = 2, 3, 4$;

$$\begin{bmatrix} \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} \widetilde{Q}_{22} & \widehat{a}_{12} \\ \widetilde{Q}_{23} & \widehat{a}_{13} \\ \widetilde{Q}_{24} & \widehat{a}_{14} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} \widetilde{Q}_{23} \\ \widetilde{Q}_{33} \\ \widetilde{Q}_{34} \end{bmatrix}, \qquad \begin{bmatrix} \widetilde{\eta}_2 \\ \widetilde{\eta}_3 \\ \widetilde{\eta}_4 \end{bmatrix} = \begin{bmatrix} \widetilde{Q}_{22} & \widehat{a}_{12} \\ \widetilde{Q}_{23} & \widehat{a}_{13} \\ \widetilde{Q}_{24} & \widehat{a}_{14} \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} \widetilde{Q}_{24} \\ \widetilde{Q}_{34} \\ \widetilde{Q}_{44} \end{bmatrix},$$

$$\begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} \widetilde{Q}_{22} & \widehat{a}_{12} \\ \widetilde{Q}_{23} & \widehat{a}_{13} \\ \widetilde{Q}_{24} & \widehat{a}_{14} \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} - c'_{01} \begin{bmatrix} \widetilde{b}_2 \\ \widetilde{b}_3 \\ \widetilde{b}_4 \end{bmatrix}, \qquad \begin{bmatrix} \widetilde{\lambda}_2 \\ \widetilde{\lambda}_3 \\ \widetilde{\lambda}_4 \end{bmatrix} = \begin{bmatrix} \widetilde{Q}_{22} & \widehat{a}_{12} \\ \widetilde{Q}_{23} & \widehat{a}_{13} \\ \widetilde{Q}_{24} & \widehat{a}_{14} \end{bmatrix} \begin{bmatrix} a_7 \\ a_8 \end{bmatrix} - c'_{02} \begin{bmatrix} \widetilde{b}_2 \\ \widetilde{b}_3 \\ \widetilde{b}_4 \end{bmatrix}.$$

When the cylinder is under *plane deformation* and *isothermal* condition as in Lekhnitskii (1981), the constants $\varepsilon = \vartheta = c'_{01} = c'_{02} = 0$ so that $a_1 = a_2 = a_3 = a_4 = 0$. For an orthotropic material, the constants $\widehat{a}_{14} = \widetilde{Q}_{24} = \widetilde{Q}_{34} = \widetilde{b}_4 = 0$ in addition. With these reductions, the stress has been checked to agree with that given by Lekhnitskii for the inhomogeneous cylinder under internal and external pressure. In his monograph Lekhnitskii did not present the solution for the displacement or explicit expressions for the stress in the cylinder under torsion and extension.

It should be noted that the plane deformation is not possible without applying adequate axial force and torque at the ends. The axisymmetric deformation and stresses involves ε and ϑ, but not A and B. In general, ε and ϑ cannot be set to zero in advance—the expressions for $\varepsilon = \varepsilon(p, P_z, M_t)$, $\vartheta = \vartheta(p, P_z, M_t)$ must be determined through Eqs. (28) and (29) by requiring that the stress resultants at the ends reduce to an axial force and a torque. The end conditions (30) and (31) are satisfied identically for axisymmetric responses. In order that the cylinder is in the plane deformation state (such that $\varepsilon = \vartheta = 0$), appropriate axial force and torque must be applied at the ends.

The foregoing solution is valid for the inhomogeneous, cylindrically anisotropic hollow cylinder subjected to radial temperature changes, uniform surface tractions, an axial force and a torque. On specifying $M_t = p_1 = p_2 = 0$ for extension by an axial force, $P_z = p_1 = p_2 = 0$ for torsion by a torque, and $M_t = P_z = 0$ for radial expansion or contraction by internal and external pressure, there follows the solutions for various loading cases.

It is clear by now that extension, torsion, and radial expansion or contraction of a cylindrically anisotropic cylinder interact. When the cylindrically anisotropic cylinder is subjected to an axial force, it exhibits not only axial extension and radial deformation but also warping and twisting. When subjected to a torque, it exhibits axial extension, radial deformation as well as warping and twisting of the cross-section. When subjected to internal, external pressure and a radially temperature change, the cross-section warps and twists as well. Coupling of extension and torsion does not occur in the inhomogeneous cylinder of orthotropic, transversely isotropic or isotropic materials.

When setting $k = m = 0$, the solution reduces to that of a homogeneous cylinder. It can be shown that when a homogeneous, cylindrically anisotropic cylinder is subjected to internal and external pressure and a radial temperature change, the maximum hoop stress σ_θ occurs at the inner surface if $c_{22} \leqslant c_{11}$, as in the case for the homogeneous isotropic material. This is not true if $c_{22} > c_{11}$—the location of the maximum hoop stress depends on the ratio of the inner and outer radii, the applied pressure as well as the material properties. A similar behavior was observed by Horgan and Baxter (1996) in the case of a homogeneous, cylindrically orthotropic material under isothermal condition. The situation is more complicated in case the material is inhomogeneous as the parameters k and m enter the picture. The maximum radial and hoop stresses in general do not occur at the inner surface of a FG cylinder.

4.2. Solid cylinder

The solutions of Eqs. (58) and (59) that satisfies Eq. (24) are trivial ones, giving $\sigma_{r\theta} = \sigma_{rz} = 0$, and $v = cr$, $w = 0$ which represent rigid body displacements as expected. The circumferential and axial displacements in the solid cylinder under axisymmetric thermomechanical loading are simply

$$u_\theta = \vartheta rz, \qquad u_z = \varepsilon z. \tag{73}$$

The radial displacement and stress are determined by solving Eq. (57) with $c'_{02} = 0$, yielding

$$\begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} = \begin{bmatrix} r^{\mu_1} & r^{\mu_2} \\ \widetilde{a}_1 r^{\mu_1+k-1} & \widetilde{a}_2 r^{\mu_2+k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \varepsilon \begin{bmatrix} a_1 r \\ a_2 r^k \end{bmatrix} + \vartheta \begin{bmatrix} a_3 r^2 \\ a_4 r^{k+1} \end{bmatrix} + \begin{bmatrix} a_5 r \\ a_6 r^k \end{bmatrix}, \tag{74}$$

where μ_1 and μ_2 are given by Eq. (68), c_1 and c_2 are constants to be determined.

The solution of a solid cylinder cannot be obtained from that of the hollow cylinder by letting $r_1 \to 0$. For a solid cylinder the displacements and stresses must remain finite at $r = 0$. It follows that the terms with negative power of r should be excluded from the solution. Examining Eq. (68), we find that $\mathrm{Re}(\mu_2)$ and $\mathrm{Re}(\mu_2 + k - 1)$ are always negative for $k \geqslant 0$. Hence we must set $c_2 = 0$. On determining c_1 from the boundary condition $(\sigma_r)_{r=r_2} = p_2$, the radial displacement and stress are found to be

$$\begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} = c_1 \begin{bmatrix} r^{\mu_1} \\ \widetilde{a}_1 r^{\mu_1+k-1} \end{bmatrix} + \varepsilon \begin{bmatrix} a_1 r \\ a_2 r^k \end{bmatrix} + \vartheta \begin{bmatrix} a_3 r^2 \\ a_4 r^{k+1} \end{bmatrix} + \begin{bmatrix} a_5 r \\ a_6 r^k \end{bmatrix}, \tag{75}$$

where $c_1 = (p_2 - \varepsilon a_2 r_2^k - \vartheta a_4 r_2^{k+1} - a_6 r_2^k)/(\widetilde{a}_1 r_2^{\mu_1+k-1})$.

The in-surface stresses are

$$\begin{bmatrix} \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \end{bmatrix} = c_1 \begin{bmatrix} \gamma_2(\mu_1) \\ \gamma_3(\mu_1) \\ \gamma_4(\mu_1) \end{bmatrix} r^{\mu_1+k-1} + \varepsilon \begin{bmatrix} \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} r^k + \vartheta \begin{bmatrix} \widetilde{\eta}_2 \\ \widetilde{\eta}_3 \\ \widetilde{\eta}_4 \end{bmatrix} r^{k+1} + \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} r^k, \tag{76}$$

where the notations are the same as defined in Eq. (72). The constants ε and ϑ are determined through Eqs. (28) and (29).

As in a hollow cylinder, extension, torsion and radial expansion or contraction interact. In Eqs. (75) and (76) the terms of r^k and r^{k+1} are the stress due to the combined action of a radial temperature change, an axial force and a torque, they are always finite for $k \geqslant 0$. The terms of r^{μ_1} and r^{μ_1+k-1} are the stress due to external pressure, they are finite when $\mu_1 + k - 1 \geqslant 0$. A peculiar situation arises when $\mu_1 + k - 1 < 0$; the stress is *singular* at $r = 0$. Such abnormality does not occur in a homogeneous isotropic cylinder. As is well known, when a homogeneous, isotropic solid cylinder is subjected to external pressure p, the stress field is uniform with the radial and hoop stresses equal to p everywhere; whereas in the cylinder with a pin hole the maximum hoop stress occurs at the center and has the value of $2p$. For an inhomogeneous, cylindrically anisotropic cylinder it can be shown that $\mu_1 + k - 1 > 0$ if $k \geqslant (1 - a_{22}a_{11}^{-1})/(1 - a_{12}a_{11}^{-1})$. This condition is always satisfied in the case of an inhomogeneous isotropic cylinder since $a_{22} = a_{11} > a_{12}$ and $k > 0$. Thus, no stress singularities arise in an inhomogeneous isotropic cylinder. When the material is cylindrically anisotropic, the condition is not always satisfied by the elastic constants. Stress singularities do not arise when $a_{22}/a_{11} \geqslant 1$, but when $a_{22}/a_{11} < 1$, the external pressure gives rise to stress singularities at $r = 0$ for certain FG cylinders in which $k < (1 - a_{22}a_{11}^{-1})/(1 - a_{12}a_{11}^{-1})$.

5. Exact solutions for rotating cylinders

When the cylinder is rotating at a constant angular velocity ω about the z-axis, the centrifugal force constitutes a body force $R = \rho r\omega^2$ in the radial direction, where $\rho = \rho(r)$ is the radially dependent mass density of the material.

Insertion of R in Eq. (39) gives

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = \begin{bmatrix} -a_{12}a_{11}^{-1} & r^{-k}a_{11}^{-1} \\ r^k\widetilde{Q}_{22} & a_{12}a_{11}^{-1} \end{bmatrix}\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} + \varepsilon\begin{bmatrix} -a_{13}a_{11}^{-1}r \\ \widetilde{Q}_{23}r^{k+1} \end{bmatrix} + \vartheta\begin{bmatrix} 0 \\ \widetilde{Q}_{24}r^{k+2} \end{bmatrix} - (c'_{01} + c'_{02}r^{-m})\begin{bmatrix} -b_1a_{11}^{-1}r \\ \widetilde{b}_2r^{k+1} \end{bmatrix} - r^2\begin{bmatrix} 0 \\ \rho(r)\omega^2 \end{bmatrix}, \tag{77}$$

for a hollow cylinder. The other equations are the same as Eqs. (58) and (59). It is easily shown that the solutions of Eqs. (58) and (59) along with the traction-free boundary conditions are trivial ones, resulting in $\sigma_{r\theta} = \sigma_{rz} = v = w = 0$. Thus, the circumferential and axial displacements in the rotating cylinder are simply

$$u_\theta = \vartheta rz, \quad u_z = \varepsilon z. \tag{78}$$

The homogeneous solution of Eq. (77) is the same as Eq. (69). In addition to the particular solution (70), the one due to the centrifugal force can be obtained for a specific variation of the mass density. Suppose $\rho(r) = \rho_0 r^\eta$, where ρ_0 is a given positive constant, η is a real number. The additional particular solution is

$$\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}_p = \begin{bmatrix} \varkappa_1\rho_0\omega^2r^{\eta+2-k} \\ \varkappa_2\rho_0\omega^2r^{\eta+2} \end{bmatrix}, \tag{79}$$

where $\varkappa_1 = a_{11}^{-1}/\Delta$, $\varkappa_2 = (\eta + 2 - k + a_{12}a_{11}^{-1})/\Delta$, $\Delta = (a_{22} - ka_{12})a_{11}^{-1} - (\eta+2)(\eta+2-k)$.

It follows from the complete solution of Eq. (77) that

$$\begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} = \begin{bmatrix} r^{\mu_1} & r^{\mu_2} \\ \widetilde{a}_1r^{\mu_1+k-1} & \widetilde{a}_2r^{\mu_2+k-1} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \varepsilon\begin{bmatrix} a_1r \\ a_2r^k \end{bmatrix} + \vartheta\begin{bmatrix} a_3r^2 \\ a_4r^{k+1} \end{bmatrix} + \begin{bmatrix} a_5r \\ a_6r^k \end{bmatrix} + \begin{bmatrix} a_7r^{1-m} \\ a_8r^{k-m} \end{bmatrix} + \begin{bmatrix} \varkappa_1\rho_0\omega^2r^{\eta+2-k} \\ \varkappa_2\rho_0\omega^2r^{\eta+1} \end{bmatrix}, \tag{80}$$

where c_1 and c_2 are undetermined constants.

For a hollow cylinder the boundary conditions are

$$(\sigma_r)_{r=r_1} = 0, \qquad (\sigma_r)_{r=r_2} = 0. \tag{81}$$

Imposing Eq. (81) on Eq. (80) yields

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} \widetilde{a}_1r_1^{\mu_1-1} & \widetilde{a}_2r_1^{\mu_2-1} \\ \widetilde{a}_1r_2^{\mu_1-1} & \widetilde{a}_2r_2^{\mu_2-1} \end{bmatrix}^{-1}\begin{bmatrix} \varkappa_2\rho_0\omega^2r_1^{\eta+1-k} + \varepsilon a_2 + \vartheta a_4r_1 + a_6 + a_8r_1^{-m} \\ \varkappa_2\rho_0\omega^2r_2^{\eta+1-k} + \varepsilon a_2 + \vartheta a_4r_2 + a_6 + a_8r_2^{-m} \end{bmatrix}. \tag{82}$$

The in-surface stresses are

$$\begin{bmatrix} \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} \gamma_2(\mu_1)r^{\mu_1+k-1} & \gamma_2(\mu_2)r^{\mu_2+k-1} \\ \gamma_3(\mu_1)r^{\mu_1+k-1} & \gamma_3(\mu_2)r^{\mu_2+k-1} \\ \gamma_4(\mu_1)r^{\mu_1+k-1} & \gamma_4(\mu_2)r^{\mu_2+k-1} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \varepsilon\begin{bmatrix} \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}r^k + \vartheta\begin{bmatrix} \widetilde{\eta}_2 \\ \widetilde{\eta}_3 \\ \widetilde{\eta}_4 \end{bmatrix}r^{k+1} + \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}r^k + \begin{bmatrix} \widetilde{\lambda}_2 \\ \widetilde{\lambda}_3 \\ \widetilde{\lambda}_4 \end{bmatrix}r^{k-m} + \rho_0\omega^2\begin{bmatrix} \widetilde{\varkappa}_2 \\ \widetilde{\varkappa}_3 \\ \widetilde{\varkappa}_4 \end{bmatrix}r^{\eta+1}, \tag{83}$$

where $\widetilde{\varkappa}_j = \varkappa_1\widetilde{Q}_{2j} + \varkappa_2a_{1j}a_{11}^{-1}$, the other notations are the same as defined in Eq. (72). The constants ε and ϑ are determined through Eqs. (28) and (29) with $P_z = M_t = 0$.

For a solid cylinder the displacements and stresses must remain finite at $r = 0$. Hence we must set $c_2 = 0$ in order to exclude the terms with negative power of r from the solution. On determining c_1 using the boundary condition $(\sigma_r)_{r=r_2} = 0$, we obtain

$$\begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} = c_1 \begin{bmatrix} r^{\mu_1} \\ \widetilde{a}_1 r^{\mu_1+k-1} \end{bmatrix} + \begin{bmatrix} a_1 r \\ a_2 r^k \end{bmatrix} + \begin{bmatrix} \varkappa_1 \rho_0 \omega^2 r^{\eta+2-k} \\ \varkappa_2 \rho_0 \omega^2 r^{\eta+1} \end{bmatrix}, \tag{84}$$

where $c_1 = -\widetilde{a}_1^{-1}(\varkappa_2 \rho_0 \omega^2 r_2^{\eta+1} + a_2 r_2^k) r_2^{\mu_1+k-1}$.

The in-surface stresses for a solid cylinder are

$$\begin{bmatrix} \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \end{bmatrix} = c_1 \begin{bmatrix} \gamma_2(\mu_1) \\ \gamma_3(\mu_1) \\ \gamma_4(\mu_1) \end{bmatrix} r^{\mu_1+k-1} + \varepsilon \begin{bmatrix} \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} r^k + \vartheta \begin{bmatrix} \widetilde{\eta}_2 \\ \widetilde{\eta}_3 \\ \widetilde{\eta}_4 \end{bmatrix} r^{k+1} + \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} r^k + \rho_0 \omega^2 \begin{bmatrix} \widetilde{\varkappa}_2 \\ \widetilde{\varkappa}_3 \\ \widetilde{\varkappa}_4 \end{bmatrix} r^{\eta+1}, \tag{85}$$

in which the constants ε and ϑ are determined through Eqs. (28) and (29) with $P_z = M_t = 0$.

The solution indicates that torsion and warping as well as extension and radial expansion occur in the rotating cylinder of a cylindrically anisotropic material. In the case of orthotropy, transverse isotropy or isotropy, torsion and warping do not occur.

It can be shown that the maximum radial and hoop stresses generally do not occur at the inner surface of an inhomogeneous hollow cylinder, or at the center of an inhomogeneous solid cylinder, in contrast to the homogeneous isotropic material. Similar conclusion has been reached by Horgan and Chan (1999) for an inhomogeneous isotropic rotating cylinder. The location of the maximum radial and hoop stress depends on the radius, the rotating speed, the mass density distribution as well as the material properties of the cylinder. The shear stress $\sigma_{\theta z}$ is non-zero in a cylindrically anisotropic rotating cylinder. It vanishes in the case of orthotropy, transverse isotropy or isotropy.

6. Closure

The foregoing analysis provides a systematic approach in the state space setting for FG anisotropic cylinders subjected to thermomechanical loading. The general solution is expressed in a matrix form. Exact solutions have been obtained for power law variations of the thermoelastic constants of the cylinder under extension, torsion, shearing, pressuring and temperature changes.

Considerations of several issues worthy further study are in order. The power law dependence of the moduli on position may not be realistic. The inhomogeneity in the FGM are usually resulted from the relative concentrations of the constituent materials. The elastic constants of a FGM are not necessarily independent as those of a homogeneous material. This restricts the property variation of a FGM. As such, the dependence of the elastic moduli on the position may not be independently assumed. In this regard, it is essential to evaluate the effective properties of the FGM in terms of the properties, volume fraction and spatial distribution of its constituents to establish the limitations on the FGM property variation.

In case the material properties of a FG cylinder are radially dependent but do not vary according to a power law distribution, one encounters in the analysis a system of differential equations with *variable* coefficients which is not easy to deal with analytically. An exact solution is out of the question. One must turn to numerical solution. An advantage of working in the state space framework is that the state equation is a standard linear system of first-order ordinary differential equations in numerical analysis. Yet it is not readily solvable numerically because one faces here a two point boundary value problem that requires considerably more effort to solve than does an initial value problem (Press et al., 1992). There are no known algorithms which a priori guarantee successful numerical solution of a two point boundary value problem. The Frobenius method of power series may be employed, but the solution is tedious and convergence of the power series solution is difficult to assess. An alternative way of dealing with a general type of radially dependent FGM is to model the inhomogeneous cylinder by a coaxial multilayered cylinder composed of fictitious layers of different materials. Then the problem can be treated by the state space approach in conjunction with the transfer matrix (Tarn and Wang, 2001a,b). The approach is effective in thermoelastic analysis of multilayered anisotropic cylinders. It requires only systematic matrix operation without recourse to a layerwise treatment. In essence, the modeling is to approximate mathematically the continuously varying properties of the FGM by piecewise constant functions. The accuracy of the approximation naturally depends on the number of fictitious layers taken, but whether the number is large or small is not a major problem.

When the cylinder is subjected to bending moments at the ends, the deformation and stress can be derived from the $n = \pm 1$ terms in the complex Fourier series (37). The governing equations (40) and (41) have to be solved along with the traction-free surface boundary conditions and the end conditions that require that the stress resultants over the cross-section reduce to the prescribed bending moments. To obtain the exact solution for the bending problem it is necessary to determine the six roots of Eq. (45) in closed forms. This is easily done numerically but rather difficult analytically. In this circumstance, it is expedient to determine the roots numerically and then carry on the analysis.

The present solutions have been obtained on the basis of stationary problems of thermoelasticity. The approach can be extended to transient thermal stress analysis and more general problems of coupled thermoelasticity. Relevant studies are currently underway.

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Appendix A

The notations used in Eqs. (39)–(42) are defined by

$$\boldsymbol{\phi}_{u0} = [-\hat{a}_{13} \quad 0 \quad 0]^{\mathrm{T}}, \qquad \boldsymbol{\phi}_{s0} = [\tilde{Q}_{23} \quad 0 \quad 0]^{\mathrm{T}}, \qquad \boldsymbol{\varphi}_{s0} = [\tilde{Q}_{24} \quad 0 \quad 0]^{\mathrm{T}},$$

$$\boldsymbol{\psi}_{u0} = [b_1 a_{11}^{-1} \quad 0 \quad 0]^{\mathrm{T}}, \qquad \boldsymbol{\psi}_{s0} = [-\tilde{b}_2 \quad 0 \quad 0]^{\mathrm{T}}, \qquad \mathbf{R} = [R \quad \Theta \quad 0]^{\mathrm{T}},$$

$$\begin{bmatrix} \tilde{s}_{55} & \tilde{s}_{56} \\ \tilde{s}_{56} & \tilde{s}_{66} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} a_{66} & -a_{56} \\ -a_{56} & a_{55} \end{bmatrix}, \qquad \Delta = a_{55}a_{66} - a_{56}^2,$$

$$\hat{a}_{ij} = a_{ij}a_{11}^{-1}, \qquad \tilde{b}_i = b_i - b_1\hat{a}_{1i}, \qquad \tilde{Q}_{ij} = a_{ij} - a_{1i}\hat{a}_{1j},$$

$$\mathbf{N}_{01} = \begin{bmatrix} -\hat{a}_{12} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{N}_{02} = \begin{bmatrix} a_{11}^{-1} & 0 & 0 \\ 0 & \tilde{s}_{55} & \tilde{s}_{56} \\ 0 & \tilde{s}_{56} & \tilde{s}_{66} \end{bmatrix}, \qquad \mathbf{N}_{03} = \begin{bmatrix} \tilde{Q}_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\phi}_{u1} = \boldsymbol{\phi}_{u0}, \qquad \boldsymbol{\psi}_{u1} = \boldsymbol{\psi}_{u0}, \qquad \boldsymbol{\phi}_{s1} = [\tilde{Q}_{23} \quad -\mathrm{i}\tilde{Q}_{23} \quad -\mathrm{i}\tilde{Q}_{34}]^{\mathrm{T}},$$

$$\boldsymbol{\psi}_{s1} = [-\tilde{b}_2 \quad \mathrm{i}\tilde{b}_2 \quad \mathrm{i}\tilde{b}_4]^{\mathrm{T}}, \qquad \mathbf{N}_{12} = \mathbf{N}_{02},$$

$$\mathbf{N}_{11} = \begin{bmatrix} -\widehat{a}_{12} & -\mathrm{i}\widehat{a}_{12} & -\mathrm{i}\widehat{a}_{14} \\ -\mathrm{i} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{N}_{13} = \overline{\mathbf{N}}_{13}^{\mathrm{T}} = \begin{bmatrix} \widetilde{Q}_{22} & \mathrm{i}\widetilde{Q}_{22} & \mathrm{i}\widetilde{Q}_{24} \\ -\mathrm{i}\widetilde{Q}_{22} & \widetilde{Q}_{22} & \widetilde{Q}_{24} \\ -\mathrm{i}\widetilde{Q}_{24} & \widetilde{Q}_{24} & \widetilde{Q}_{44} \end{bmatrix},$$

$$\boldsymbol{\psi}_{un} = \boldsymbol{\psi}_{u0}, \qquad \boldsymbol{\psi}_{sn} = \begin{bmatrix} -\widetilde{b}_2 & \mathrm{i}n\widetilde{b}_2 & \mathrm{i}n\widetilde{b}_4 \end{bmatrix}^{\mathrm{T}}, \qquad \mathbf{N}_{n2} = \mathbf{N}_{02},$$

$$\mathbf{N}_{n1} = \begin{bmatrix} -\widehat{a}_{12} & -\mathrm{i}n\widehat{a}_{12} & -\mathrm{i}n\widehat{a}_{14} \\ -\mathrm{i}n & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{N}_{n3} = \overline{\mathbf{N}}_{n3}^{\mathrm{T}} = \begin{bmatrix} \widetilde{Q}_{22} & \mathrm{i}n\widetilde{Q}_{22} & \mathrm{i}n\widetilde{Q}_{24} \\ -\mathrm{i}n\widetilde{Q}_{22} & n^2\widetilde{Q}_{22} & n^2\widetilde{Q}_{24} \\ -\mathrm{i}n\widetilde{Q}_{24} & n^2\widetilde{Q}_{24} & n^2\widetilde{Q}_{44} \end{bmatrix}.$$

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